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Wigner–Souriau translations and Lorentz symmetry of chiral fermions

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ABSTRACT

Chiral fermions can be embedded into Souriau's massless spinning particle model by "enslaving" the spin, viewed as a gauge constraint. The latter is not invariant under Lorentz boosts; spin enslavement can be restored, however, by a Wigner–Souriau (WS) translation, analogous to a compensating gauge transformation. The combined transformation is precisely the recently uncovered twisted boost, which we now extend to finite transformations. WS-translations are identified with the stability group of a motion acting on the right on the Poincaré group, whereas the natural Poincaré action corresponds to action on the left.

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1. Introduction

Semiclassical chiral fermions of spin $\frac{1}{2}$ can be described by the phase-space action

$$S=\int\left((\boldsymbol{p}+e\boldsymbol{A})\cdot\frac{d\boldsymbol{x}}{dt}-h-\boldsymbol{a}\cdot\frac{d\boldsymbol{p}}{dt}\right)dt,\quad h=|\boldsymbol{p}|+e\phi(\boldsymbol{x}),\quad(1.1)$$

where $\boldsymbol{a}(\boldsymbol{p})$ is a vector potential for the "Berry monopole" in $\boldsymbol{p}$-space, $\nabla_{\boldsymbol{p}}\times\boldsymbol{a}=\frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}$, $\widehat{\boldsymbol{p}}=\boldsymbol{p}|\boldsymbol{p}|$, $\boldsymbol{p}\neq 0$. Here $\boldsymbol{A}(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ are [static] vector and scalar potentials and e is the electric charge [1–6]. A distinctive feature is that spin is "enslaved" to the momentum, i.e.,

$$\boldsymbol{s}=\frac{1}{2}\widehat{\boldsymbol{p}}.\quad(1.2)$$

An intriguing aspect of the model is its *lack of manifest Lorentz symmetry*. Recently [5], it was shown, though, that modifying the dispersion relation in (1.1) as $h=|\boldsymbol{p}|+e\phi(\boldsymbol{x})+\frac{\widehat{\boldsymbol{p}}\cdot\boldsymbol{B}}{2|\boldsymbol{p}|}$ yields a theory which is covariant w.r.t. Lorentz transformations. Turning off the external field, their expression # (6) reduces to

$$\delta\boldsymbol{x}=\boldsymbol{\beta}t+\boldsymbol{\beta}\times\frac{\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|},\qquad\delta\boldsymbol{p}=|\boldsymbol{p}|\boldsymbol{\beta},\qquad\delta t=\boldsymbol{\beta}\cdot\boldsymbol{x},\quad(1.3)$$

where $\boldsymbol{\beta}$ is an infinitesimal Lorentz boost. This formula has also been found, independently [6], by relating the chiral fermion to Souriau's model of a relativistic massless spinning particle [6,7].

The chiral and the Souriau systems [7] have seemingly different degrees of freedom: the first one has "enslaved" spin, while the latter has "unchained" spin represented by a vector $\boldsymbol{s}$, which may not satisfy (1.2). However, the freedom specific to the massless case of applying what we call a *Wigner–Souriau* (WS) *translation*, Eq. (2.7) below,[1] allows one to eliminate the additional degrees of freedom of the Souriau framework, so that the two systems have identical spaces of motions [6,7]. This enabled us to "*export*" the natural Lorentz symmetry to the chiral system, yielding (1.3).

In this Note we derive (1.3) in another, rather more intuitive way, namely by *embedding* the chiral fermion model directly into the Souriau model, namely by viewing spin enslavement, (1.2), as a *gauge condition* with WS translations and viewed as gauge transformations. Then our clue is that a natural Lorentz boost does not leave the constraint (1.2) invariant; the "gauge" condition (1.2) can, however, be restored by a WS translation, yielding, once again, (1.3). This is reminiscent of what happens in *gauge theories*, where a space–time transformation can be a symmetry when the variation of the vector potential can be compensated by a suitable gauge transformation [12].

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[1] Wigner–Souriau translations (called "Z-shifts" in [6]) were known to Wigner [8] and to Penrose [9]. Their use in the semiclassical framework was advocated by Souriau [7]. They were overlooked by most present authors with the notable exception of Stone et al. [4], who also suggested viewing them as gauge transformations.

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Further insight is gained in Section 5 which clarifies the geometry hidden behind: while the natural Poincaré action corresponds the left-action of the Poincaré group on itself, WS translations correspond to the right-action of the stability group of a motion.

We conclude our Letter by some remarks about non-commutativity.

2. Symplectic description of the chiral and the massless spinning models

Both the chiral and the Souriau models can conveniently be described within Souriau's framework [6,7], where the classical motions are identified with curves or surfaces in some evolution space, and are tangent to the kernel of a closed two-form; see [6] for details. We limit our considerations to the free case; coupling to external fields has been discussed in the literature, see [6,13].

We first consider the [free] chiral model (1.1). It has been shown [6] that the associated variational problem admits an alternative geometric formulation. To that end, we introduce the seven-dimensional evolution space $V^7 = T(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ described by triples $(\boldsymbol{x}, \boldsymbol{p}, t)$, and endow it with the two-form σ defined by

$$\sigma = \omega - dh \wedge dt \quad \text{where } \omega = dp_i \wedge dx^i - \frac{1}{4|\boldsymbol{p}|^3}\epsilon^{ijk} p_i \, dp_j \wedge dp_k,$$
$$h = |\boldsymbol{p}|. \tag{2.1}$$

The two-forms ω and σ are closed, since $\nabla_{\boldsymbol{p}} \cdot \boldsymbol{b} = 0$. The kernel of σ defines an integrable distribution, whose leaves [integral manifolds] can be viewed as generalized solutions of the variational problem. Here, the kernel is one-dimensional and a curve $(\boldsymbol{x}(t), \boldsymbol{p}(t), t)$ is tangent to it iff the equations of motion,

$$\frac{d\boldsymbol{x}}{dt} = \widehat{\boldsymbol{p}}, \qquad \frac{d\boldsymbol{p}}{dt} = 0, \tag{2.2}$$

are satisfied [6]; the solution is plainly $\boldsymbol{p} = \boldsymbol{p}_0 = \text{const}$, $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \widehat{\boldsymbol{p}}t$, $\boldsymbol{x}_0 = \text{const}$, i.e., the motion is in the $\widehat{\boldsymbol{p}}$ direction with the velocity of light.

The Souriau model admits a similar description [6]. Restricting ourselves again to the free case, the evolution space here is nine-dimensional and is described by

$$V^9 = \Big\{ R, P \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) \,\Big|\, P_\mu P^\mu = 0,\ P^4 > 0,\ S_{\mu\nu}P^\nu = 0,$$
$$S_{\mu\nu}S^{\mu\nu} = \frac{1}{2} \Big\}. \tag{2.3}$$

The evolution space is endowed with the closed two-form

$$\sigma = -dP_\mu \wedge dR^\mu - 2\, dS^\mu{}_\lambda \wedge S^\lambda{}_\rho \, dS^\rho{}_\mu. \tag{2.4}$$

A $(3+1)$-decomposition can be introduced by writing, in a Lorentz frame, $R = (\boldsymbol{r}, t)$, and $P = (\boldsymbol{p}, |\boldsymbol{p}|)$ where $\boldsymbol{p} \neq 0$. The components $S_{\mu\nu}$ of the spin tensor can in turn be split into space and space–time components,

$$S_{ij} = \epsilon_{ijk} s^k, \quad S_{j4} = \Sigma_j \quad \text{where } \boldsymbol{\Sigma} = \widehat{\boldsymbol{p}} \times \boldsymbol{s}, \tag{2.5}$$

the 3-vector $\boldsymbol{s}$ being interpreted as the spin in the chosen Lorentz frame. Note that $\boldsymbol{\Sigma}$ is not an independent variable, so that a point of V^9 can be labeled by $(\boldsymbol{r}, t, \boldsymbol{p}, \boldsymbol{s})$. Then the [free] equations of motion associated with the kernel of σ in (2.4) are expressed as,[2]

$$-\boldsymbol{p} \cdot \dot{\boldsymbol{r}} + |\boldsymbol{p}|\dot{t} = 0, \qquad \dot{\boldsymbol{p}} = 0, \qquad \dot{\boldsymbol{s}} = \boldsymbol{p} \times \dot{\boldsymbol{r}}, \tag{2.6}$$

and can be deduced from Eq. (3.6) in [6]. A particular solution of (2.6) is obtained by *embedding the chiral solution above into the spin-extended evolution space* V^9 by identifying $\boldsymbol{r}$ with $\boldsymbol{x}$ in (2.2) and completing [trivially] with a constant spin vector, $\boldsymbol{p} = \boldsymbol{p}_0 = \text{const}$, $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \widehat{\boldsymbol{p}}t$, $\boldsymbol{r}_0 = \text{const}$, $\boldsymbol{s}(t) = \boldsymbol{s}_0 = \text{const}$. A remarkable feature of Eqs. (2.6) is that, for an arbitrary 3-vector $\boldsymbol{W}$, the transformation we refer to as a *Wigner–Souriau* (WS) *translation* [4,6–9],

$$\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{W},\ t \to t + \widehat{\boldsymbol{p}} \cdot \boldsymbol{W},\ \boldsymbol{p} \to \boldsymbol{p},\ \boldsymbol{s} \to \boldsymbol{s} + \boldsymbol{p} \times \boldsymbol{W} \tag{2.7}$$

takes a solution of (2.6) into another, equivalent one. The kernel of (2.4) is invariant under WS-translations and is in fact three-dimensional, swept by the images of the embedded solutions.

The spin vector here, $\boldsymbol{s}$, is not necessarily "enslaved", i.e., may not be parallel to the momentum, $\boldsymbol{p}$. The spin constraint in (2.3) implies nevertheless that the projection of the spin onto the momentum and the perpendicular component $\boldsymbol{\Sigma}$ satisfy,

$$\boldsymbol{s} \cdot \widehat{\boldsymbol{p}} = \frac{1}{2}, \qquad \boldsymbol{\Sigma} = \widehat{\boldsymbol{p}} \times \boldsymbol{s}, \tag{2.8}$$

respectively, see (2.5). From the WS action above we infer that $\boldsymbol{\Sigma} \to \boldsymbol{\Sigma} + \widehat{\boldsymbol{p}} \times (\boldsymbol{p} \times \boldsymbol{W})$. It follows that $\boldsymbol{\Sigma}$ can be *eliminated*: choosing $\boldsymbol{W} = \boldsymbol{\Sigma}/|\boldsymbol{p}|$ carries $\boldsymbol{\Sigma}$ to zero.

The Poincaré group acts naturally on V^9, namely according to [6,7]

$$\begin{cases} \delta \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{\beta} t + \boldsymbol{\gamma}, \\ \delta t = \boldsymbol{\beta} \cdot \boldsymbol{r} + \varepsilon, \\ \delta \boldsymbol{p} = \boldsymbol{\omega} \times \boldsymbol{p} + \boldsymbol{\beta}\, |\boldsymbol{p}|, \\ \delta \boldsymbol{s} = \boldsymbol{\omega} \times \boldsymbol{s} - \boldsymbol{\beta} \times \boldsymbol{\Sigma}, \end{cases} \tag{2.9}$$

where ω, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, ε are identified with infinitesimal rotations, boosts, translations and time-translations; their action duly projects to Minkowski space–time as the natural one. In what follows, we focus our attention at boosts; WS-translations will be studied further in Section 5.

3. Embedding the chiral system into the massless spinning model

Now we embed the evolution space of the chiral model, V^7, into that of the massless spinning particle, V^9. We note first that, by (2.8), spin enslavement, (1.2), is equivalent to

$$\boldsymbol{\Sigma} = \widehat{\boldsymbol{p}} \times \boldsymbol{s} = 0, \tag{3.1}$$

which, viewed as a constraint, defines a seven-dimensional submanifold of V^9 that we parametrize with $\boldsymbol{r}, \boldsymbol{p}, t$ and denote (with a suggestive abuse of notation) still V^7. Eqs. (2.6) and (2.8) imply that $\dot{\boldsymbol{\Sigma}} = |\boldsymbol{p}|(\widehat{\boldsymbol{p}}\dot{t} - \dot{\boldsymbol{r}})$. Requiring $\boldsymbol{\Sigma} = 0$ is therefore consistent with the dynamics: the motions of the chiral system lie in the intersection of the three-dimensional characteristic leaves of V^9 with the surface V^7 defined by spin enslaving; they remain therefore motions also for the extended dynamics.[3] A chiral motion is embedded into V^9 by respecting the gauge condition (1.2), namely as $\gamma(t) = (\boldsymbol{r}(t) = \boldsymbol{x}(t), \boldsymbol{p}(t), \boldsymbol{s}(t) = \frac{1}{2}\widehat{\boldsymbol{p}}(t))$.

4. Lorentz boost actions

We consider now an infinitesimal Lorentz boost, $\boldsymbol{\beta}$. For $\boldsymbol{s} = \frac{1}{2}\widehat{\boldsymbol{p}}$ we have $\delta_{\boldsymbol{\beta}}(\boldsymbol{p} \times \boldsymbol{s}) = \frac{1}{2}\, \boldsymbol{\beta} \times \boldsymbol{p}$, which does *not* vanish in general: *spin*

[2] These equations can also be obtained in a Wess–Zumino framework [10], namely as the variational equations of the Lagrangian # (3.1) of [10]; the latter is in fact the one-form (B.1) in [6], whose exterior derivative is the two-form $d(\mu_0 \cdot g^{-1}dg)$.

[3] Alternatively, the restriction of the free two-form (2.4) of V^9 to V^7 is (2.1).

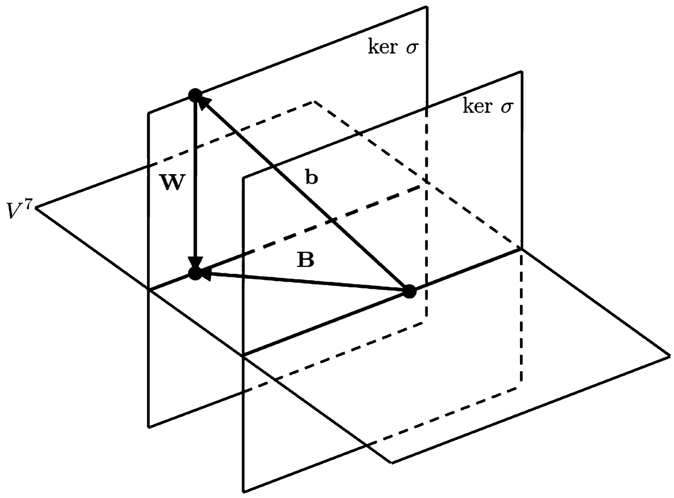

Fig. 1. Enslaving the spin amounts to embedding the chiral evolution space V^7 into that, V^9, of the massless spinning particle. A chiral motion thus becomes the intersection of V^7 with the characteristic leaf of $\ker\sigma$ (depicted as a vertical plane). Enslavement is not invariant under a (natural) boost $\boldsymbol{b}$, but a suitable compensating Wigner–Souriau translation $\boldsymbol{W}$ allows us to re-enslave the spin. The combination of the two transformations yields the twisted boost $\boldsymbol{B}$ in (1.3).

and momentum do not remain parallel even if they were so initially: *embedding the chiral system into the Souriau model through spin enslavement is not boost-invariant: natural boost symmetry is broken* by spin enslavement. However, as dictated by the analogy with gauge symmetries [12], let us apply an infinitesimal WS-translation

$$\delta_{\boldsymbol{W}}\boldsymbol{r}=\boldsymbol{W},\qquad \delta_{\boldsymbol{W}}t=\widehat{\boldsymbol{p}}\cdot\boldsymbol{W},\qquad \delta_{\boldsymbol{W}}\boldsymbol{p}=0,$$
$$\delta_{\boldsymbol{W}}\boldsymbol{s}=\boldsymbol{p}\times\boldsymbol{W}\quad\text{with } \boldsymbol{W}=\frac{\boldsymbol{\beta}\times\widehat{\boldsymbol{p}}}{2|\boldsymbol{p}|}. \tag{4.1}$$

Then $\delta_{\boldsymbol{W}}\boldsymbol{s}=\frac{1}{2}(\boldsymbol{\beta}-\widehat{\boldsymbol{p}}(\widehat{\boldsymbol{p}}\cdot\boldsymbol{\beta}))$, implying that the combined transformation $\delta=\delta_{\boldsymbol{\beta}}+\delta_{\boldsymbol{W}}$ *does preserve spin enslavement*, $\delta(\boldsymbol{p}\times\boldsymbol{s})=0$. The transformation of V^7 generated by δ is precisely the twisted boost (1.3).[4]

So far we studied infinitesimal actions only. But our strategy is valid also for finite transformations, as we show it now. Firstly, from Appendix C of [6] we infer the action of a finite boost $\boldsymbol{b}$ on the evolution space V^9, namely

$$\begin{cases}\boldsymbol{r}'=\boldsymbol{r}+(\gamma-1)(\widehat{\boldsymbol{b}}\cdot\boldsymbol{r})\widehat{\boldsymbol{b}}+\gamma t\boldsymbol{b},\\ t'=\gamma(\boldsymbol{b}\cdot\boldsymbol{r}+t),\\ \boldsymbol{p}'=\boldsymbol{p}+\gamma|\boldsymbol{p}|\boldsymbol{b}+(\gamma-1)(\widehat{\boldsymbol{b}}\cdot\boldsymbol{p})\widehat{\boldsymbol{b}},\\ \boldsymbol{s}'=\boldsymbol{s}+(\gamma-1)\widehat{\boldsymbol{b}}\times(\boldsymbol{s}\times\widehat{\boldsymbol{b}})-\gamma\boldsymbol{b}\times\boldsymbol{\Sigma},\end{cases} \tag{4.2}$$

which is consistent with the infinitesimal action (2.9). Then, starting with enslaved spin, $\boldsymbol{s}=\frac{1}{2}\widehat{\boldsymbol{p}}$, we find,

$$\boldsymbol{\Sigma}'=\frac{\boldsymbol{p}'\times\boldsymbol{s}'}{|\boldsymbol{p}'|}=\frac{1}{2}\big(\gamma^2|\boldsymbol{b}|+\big(\gamma^2-1\big)\widehat{\boldsymbol{b}}\cdot\widehat{\boldsymbol{p}}\big)\frac{\widehat{\boldsymbol{b}}\times\boldsymbol{p}}{|\boldsymbol{p}'|} \tag{4.3}$$

which does not vanish in general: spin is unchained. At last, the finite WS-translation (2.7) with $\boldsymbol{W}=\boldsymbol{\Sigma}'/|\boldsymbol{p}'|$ restores the validity of (1.2): spin is re-enslaved, $\boldsymbol{s}''=\frac{1}{2}\widehat{\boldsymbol{p}}'$. The combined transformation for finite boosts,

$$\boldsymbol{r}''=\boldsymbol{r}+\gamma t\boldsymbol{b}+(\gamma-1)(\widehat{\boldsymbol{b}}\cdot\boldsymbol{r})\widehat{\boldsymbol{b}}+\frac{\boldsymbol{\Sigma}'}{|\boldsymbol{p}'|}, \tag{4.4}$$

completed with $t''=t'$ and $\boldsymbol{p}''=\boldsymbol{p}'$ where t', $\boldsymbol{p}'$ and $\boldsymbol{\Sigma}'$ are given in (4.2) and (4.3), respectively. The infinitesimal action is (1.3) as it should be. In conclusion, Fig. 1 is *valid also for finite boosts*.

We now turn to the *space of motions* [6,7], defined as the quotient of the evolution space by the characteristic foliation of σ; we denote it by M. The equations of motion of the spin-extended system, (2.6), imply that

$$\tilde{\boldsymbol{x}}(t)=\boldsymbol{r}(t)-\widehat{\boldsymbol{p}}t+\frac{\boldsymbol{\Sigma}}{|\boldsymbol{p}|} \tag{4.5}$$

is in fact a constant of the motion, $d\tilde{\boldsymbol{x}}/dt=0$. It can be used therefore to label the motion, i.e., a characteristic leaf in V^9. The conserved momentum, $\boldsymbol{p}$, is another good coordinate set on M, which is six-dimensional, and whose points can therefore be labeled by $\tilde{\boldsymbol{x}}$ and $\boldsymbol{p}\neq 0$.

In [6], we derived the twisted boost (1.3) from the Poincaré action on the space of motions. Conversely, the latter can be obtained from our construction here. Boosting in V^9 according to (2.9) we find that the preceding terms combine to yield

$$\delta_{\boldsymbol{\beta}}\tilde{\boldsymbol{x}}=\frac{1}{2}\boldsymbol{\beta}\times\frac{\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|}-\widehat{\boldsymbol{p}}(\boldsymbol{\beta}\cdot\tilde{\boldsymbol{x}}). \tag{4.6}$$

Now WS-translations act in turn trivially on the space of motions, $\delta_{\boldsymbol{W}}\tilde{\boldsymbol{x}}=\delta_{\boldsymbol{W}}\boldsymbol{p}=0$, since they move each characteristic leave within itself. Completing (4.6) with $\delta_{\boldsymbol{\beta}}\boldsymbol{p}=|\boldsymbol{p}|\boldsymbol{\beta}$ cf. (2.9), we end up with the (boost-)action on the space of motions, – Eq. (3.19) in [6]. At last, applying (4.2) to each term in (4.5) allows us to confirm the finite action on the space of motions, Eq. # (3.24) in [6].

5. The geometry of Wigner–Souriau transformations

Do WS-translations belong to the Poincaré group? To answer this question, let us first briefly summarize Souriau's construction for "elementary systems" [meaning that the group acts symplectically and transitively] [7].

The connected component of the Poincaré group, G, can be identified with the group of matrices $g=\begin{pmatrix}L & C\\ 0 & 1\end{pmatrix}$ where L is a Lorentz transformation, $L\in H$ (the connected Lorentz group), and C a Minkowski-space vector. The Poincaré Lie algebra, $\mathfrak{g}$, is hence spanned by the matrices $X=\begin{pmatrix}\Lambda & \Gamma\\ 0 & 0\end{pmatrix}$ where the infinitesimal Lorentz transformations and translations, $\Lambda\in\mathrm{so}(3,1)$, and $\Gamma\in\mathbb{R}^{3,1}$, respectively, are such that, in a given Lorentz frame, $\Lambda=\begin{pmatrix}j(\boldsymbol{\omega}) & \boldsymbol{\beta}\\ \boldsymbol{\beta}^T & 0\end{pmatrix}$ and $\Gamma=\begin{pmatrix}\boldsymbol{\gamma}\\ \varepsilon\end{pmatrix}$ with $\boldsymbol{\omega},\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^3$ interpreted as an infinitesimal rotation, boost, and space-translation, while $\varepsilon\in\mathbb{R}$ stands for an infinitesimal time-translation. Here $j(\boldsymbol{\omega})$ is the matrix of cross product in 3-space, $j(\boldsymbol{\omega})\boldsymbol{x}=\boldsymbol{\omega}\times\boldsymbol{x}$.

The Poincaré group acts on itself *on the left*, $g\to hg$, for $h\in G$, and the quotient of G by the subgroup of Lorentz transformations, G/H, can be identified with Minkowski space–time. This left-action of G on itself projects to the natural action, infinitesimally the two upper lines in (2.9), of the Poincaré group on space–time.

The Poincaré group acts on the dual Lie algebra by the coadjoint representation, defined by $\mathrm{Coad}_g\mu\cdot X=\mu\cdot g^{-1}Xg$, for all $X\in\mathfrak{g}$. We denote here by $\mu=(M,P)\in\mathfrak{g}^*$ a "moment" of the Poincaré group, and by $\mu\cdot X=\frac{1}{2}M_{\mu\nu}\Lambda^{\mu\nu}-P_\mu\Gamma^\mu$ its contraction with $X\in\mathfrak{g}$, see [7].

Contracting the Maurer–Cartan one-form, $g^{-1}dg$, with and arbitrary fixed element μ_0 of the dual of Lie algebra, yields a real one-form on the group G; we denote by $\sigma=d(\mu_0\cdot g^{-1}dg)$ its exterior derivative. As a general fact, the characteristic leaves of the two-form σ, defined by its kernel, are identified with the classical motions [11] associated with μ_0. Indeed, the space of all motions,

[4] The order is irrelevant, since WS-shifts and boosts commute.

M, of an elementary system for the group G is interpreted by Souriau [7] as the orbit of some basepoint μ_0 under the coadjoint action, $M = \mathrm{Coad}_G\mu_0 \approx G/G_0$, where G_0 is the stability subgroup of μ_0. The coadjoint orbit, M, is hence canonically endowed with the symplectic structure defined by the Kostant–Kirillov–Souriau two-form ω, the image of σ under the projection $G \to M$.

In our case and following [7], the basepoint can readily be chosen as $\mu_0 = (M_0, P_0)$ with $M_0 = s\begin{pmatrix} j(\widehat{\boldsymbol{p}}_0) & 0 \\ 0 & 0\end{pmatrix}$ and $P_0 = |\boldsymbol{p}_0|\begin{pmatrix}\widehat{\boldsymbol{p}}_0 \\ 1\end{pmatrix}$ with $\widehat{\boldsymbol{p}}_0 = \mathbf{e}_3$; here $s = \frac{1}{2}$ is the scalar spin and the positive constant $|\boldsymbol{p}_0|$ is the energy. Then a straightforward calculation shows that the Poincaré Lie algebra element (Λ, Γ) with parameters $\boldsymbol{\omega}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3, \varepsilon \in \mathbb{R}$ leaves the chosen basepoint μ_0 invariant whenever

$$\boldsymbol{\omega} = \boldsymbol{\beta} \times \widehat{\boldsymbol{p}}_0 + \lambda\widehat{\boldsymbol{p}}_0, \qquad \boldsymbol{\gamma} = -\frac{1}{2}\frac{\boldsymbol{\beta}\times\widehat{\boldsymbol{p}}_0}{|\boldsymbol{p}_0|} + \varepsilon\,\widehat{\boldsymbol{p}}_0. \tag{5.1}$$

The stability Lie algebra, $\mathfrak{g}_0$, is therefore four-dimensional, parametrized by $(\boldsymbol{\beta}, \varepsilon, \lambda)$, where $\boldsymbol{\beta} \perp \widehat{\boldsymbol{p}}_0$, $\varepsilon \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ represents an infinitesimal rotation around $\widehat{\boldsymbol{p}}_0$. We note that the evolution space V^9 is in fact the quotient of the Poincaré group by rotations around $\widehat{\boldsymbol{p}}_0$, and (2.9) above is in fact the projection of the infinitesimal left-action of the group G to $V^9 \approx G/\mathrm{SO}(2)$, whereas the two-form $d(\mu_0 \cdot g^{-1}dg)$ projects as (2.4), as anticipated by the notation.

Remember now that the Poincaré group also acts on itself *from the right*, $g \to gh^{-1}$, for $h \in G$; its infinitesimal right-action is therefore given by matrix multiplication, $\delta_X g = -gX$, where $X = \delta h$ at $h = 1$. Choosing in particular $X \in \mathfrak{g}_0$, the action on the group reads $\delta_X\begin{pmatrix} L & C \\ 0 & 1\end{pmatrix} = \begin{pmatrix} -L\Lambda & -L\Gamma \\ 0 & 0\end{pmatrix}$ and $\Gamma = \begin{pmatrix}\boldsymbol{\gamma} \\ \varepsilon\end{pmatrix}$, where $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ are as in (5.1). This 4-parameter vector field belongs, at each point $g \in G$, to the kernel of the two-form σ; in fact, it *generates* its kernel [7]. Renaming the Poincaré translation C as $R = (\boldsymbol{r}, t)$, a space–time event, shows that the right-action on G of a vector X from the stability algebra $\mathfrak{g}_0$ yields,

$$\delta_X\boldsymbol{r} = \frac{1}{2}\frac{\boldsymbol{\beta}\times\widehat{\boldsymbol{p}}}{|\boldsymbol{p}|} - \varepsilon\,\widehat{\boldsymbol{p}} \quad\text{and}\quad \delta_X t = -\varepsilon. \tag{5.2}$$

This transformation satisfies the condition $\widehat{\boldsymbol{p}}\cdot\delta_X\boldsymbol{r} = \delta_X t$ required for an infinitesimal WS-translation; conversely, any WS-translation $\boldsymbol{W}$ is of the form (5.2), with $\boldsymbol{\beta}$ perpendicular to $\boldsymbol{p}$ and λ arbitrary. Therefore (with a slight abuse) we will refer to $\mathfrak{g}_0$ acting from the right as a WS-translations.

Comparing now (5.2) with (4.1) allows us to conclude that *while a boost $\boldsymbol{\beta}$ acting on the left unchains the spin, the latter is re-enslaved by a WS-translation with the same boost, $\boldsymbol{\beta}$, acting from the right*.

It is readily verified that the right-action of an X from the stability algebra $\mathfrak{g}_0$ acts as $\delta_X\boldsymbol{p} = 0$ and $\delta_X\boldsymbol{s} = \frac{1}{2}\,\widehat{\boldsymbol{p}}\times(\widehat{\boldsymbol{p}}\times\boldsymbol{\beta})$, consistently with the infinitesimal action. Thus, after rotations around $\widehat{\boldsymbol{p}}_0$ are factored out, the right-action of the stability subalgebra $\mathfrak{g}_0$ projects to V^9 as the WS translations. In conclusion, Wigner–Souriau translations *are* Poincaré transformations, — but which act on the *right* and not on the left as natural ones do. Projecting further down to the space of motions, M, they act trivially. The various spaces are shown, symbolically, in Fig. 2.

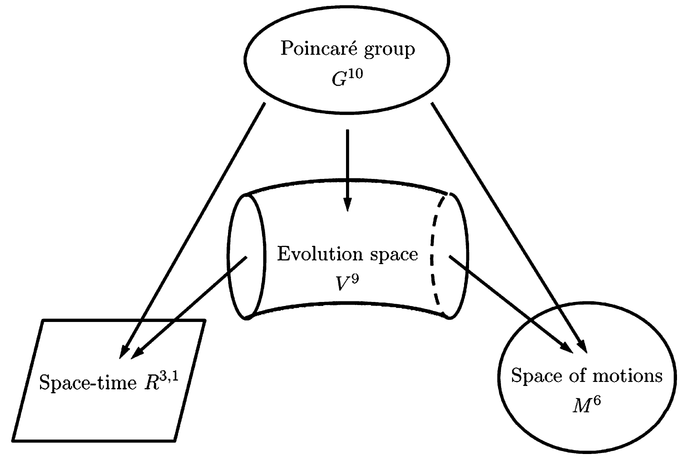

Fig. 2. The evolution space, V^9, is the quotient of the Poincaré group, G^{10} by rotations around $\widehat{\boldsymbol{p}}_0$. The space of motions is identified with a coadjoint orbit $M^6 \approx G^{10}/G_0$ of G^{10}, where G_0, the stability group of the basepoint μ_0. The left-action of G^{10} on itself projects to the natural Poincaré action on both the space of motions and Minkowski space–time, consistently with projecting first to V^9 and then to M^6 and $\mathbb{R}^{3,1}$, respectively.

6. Relation to non-commutativity

The spaces of motions of both the chiral and the Souriau systems carry the symplectic structure ω in (2.1) [with $\tilde{\boldsymbol{x}}$ replacing $\boldsymbol{x}$] [6]. Translating into Poisson bracket language, we have $\{\xi^\alpha, \xi^\beta\} = \omega^{\alpha\beta}(\xi)$ where $(\omega^{\alpha\beta}) = -(\omega_{\alpha\beta})^{-1}$ are the coefficients of the inverse of the symplectic form $\omega = \frac{1}{2}\omega_{\alpha\beta}(\xi)d\xi^\alpha\wedge d\xi^\beta$. Therefore the components of the space of motions coordinates $\tilde{\boldsymbol{x}}$ do *not to Poisson-commute*,

$$\{\tilde{x}^i, \tilde{x}^j\} = \frac{1}{2}\epsilon^{ijk}\frac{p_k}{|\boldsymbol{p}|^3}. \tag{6.1}$$

Putting $t = 0$ and $\boldsymbol{\Sigma} = 0$ into (4.5) shows that the chiral coordinate $\boldsymbol{x}$ satisfies the same commutation relation. Non-commutativity is thus a consequence of spin, cf. [15–18]; chiral fermions provide just another example of non-commutative mechanics.

It is worth stressing that the non-commutativity is *unrelated* to WS translations. Non-commutativity arises in fact due to the twisted symplectic structure of the *space of motions* (modeled by the phase space); but the WS translations act on the evolution space *before projecting, trivially*, to the space of motions.

In the operator context a *Bopp-shift* transforms the commutation relations to canonical ones [19]. In the (semi)classical context we are interested in it amounts to introducing new, canonical coordinates, eliminating spatial non-commutativity altogether. Now Darboux's theorem [20] guarantees that this is always possible *locally*. In detail, we redefine the coordinates as

$$\widetilde{\boldsymbol{X}} = \tilde{\boldsymbol{x}} + \boldsymbol{a}(\boldsymbol{p}), \qquad \widetilde{\boldsymbol{P}} = \boldsymbol{p}, \tag{6.2}$$

where $\boldsymbol{a}(\boldsymbol{p})$ is a "Berry" vector potential. Then, in terms of $\widetilde{\boldsymbol{P}}$ and $\widetilde{\boldsymbol{X}}$, the symplectic form ω in (2.1) has the canonical expression $\omega = d\widetilde{P}_i \wedge d\widetilde{X}^i$, so that $\{\widetilde{X}^i, \widetilde{X}^j\} = \{\widetilde{P}_i, \widetilde{P}_j\} = 0$ and $\{\widetilde{P}_i, \widetilde{X}^j\} = \delta_i^j$. Spin seems thus eliminated. This is *not* possible globally, though, since any $\boldsymbol{a}$ is singular along a Dirac string, as it is well-known.

The variation of the Berry vector potential under a boost $\boldsymbol{\beta} \in \mathbb{R}^3$ is then found to be $\delta_{\boldsymbol{\beta}}\boldsymbol{a} = -\boldsymbol{\beta}\times\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} - (\boldsymbol{a}\cdot\boldsymbol{\beta})\,\hat{\boldsymbol{p}} + \nabla_{\boldsymbol{p}}(|\boldsymbol{p}|\boldsymbol{\beta}\cdot\boldsymbol{a})$. It follows that, in terms of the Darboux coordinates (6.2), the infinitesimal action of boosts is given by

$$\delta_{\boldsymbol{\beta}}\widetilde{\boldsymbol{X}} = -(\boldsymbol{\beta}\cdot\widetilde{\boldsymbol{X}})\frac{\widetilde{\boldsymbol{P}}}{|\widetilde{\boldsymbol{P}}|} + \nabla_{\widetilde{\boldsymbol{P}}}\big(|\widetilde{\boldsymbol{P}}|\boldsymbol{\beta}\cdot\boldsymbol{a}\big), \qquad \delta_{\boldsymbol{\beta}}\widetilde{\boldsymbol{P}} = \boldsymbol{\beta}|\widetilde{\boldsymbol{P}}|. \tag{6.3}$$

Redefining analogously the chiral coordinate as $\boldsymbol{X} = \boldsymbol{x} + \boldsymbol{a}$ and $\boldsymbol{P} = \boldsymbol{p}$ allows us to infer

$$\begin{aligned}\delta_{\boldsymbol{\beta}}\boldsymbol{X} &= \boldsymbol{\beta}t - (\boldsymbol{\beta}\cdot\boldsymbol{a})\hat{\boldsymbol{P}} + \nabla_{\boldsymbol{P}}\big(|\boldsymbol{P}|\,\boldsymbol{\beta}\cdot\boldsymbol{a}\big) = \boldsymbol{\beta}t + |\boldsymbol{P}|\nabla_{\boldsymbol{P}}(\boldsymbol{\beta}\cdot\boldsymbol{a}),\\ \delta_{\boldsymbol{\beta}}\boldsymbol{P} &= \boldsymbol{\beta}|\boldsymbol{P}|,\end{aligned} \tag{6.4}$$

which is reminiscent of but still different from the natural action (2.9).

7. Conclusion

In this Letter, we re-derived the twisted Lorentz symmetry (1.3) of chiral fermions by embedding the theory of Refs. [1–5] into Souriau's massless spinning model [6,7] by spin enslavement, (1.2), viewed as a gauge fixing. The latter is not boost invariant, but enslavement can be restored by a suitable compensating WS-translation, which is analogous to a gauge transformation [12]. Our formula (4.4) extends (1.3) to finite transformations.

The motions of the extended model are not mere curves but 3-planes, swept by WS-translations. The massless relativistic particle is delocalized [4,6–10] and behaves rather as a *3-brane*.

Coupling to an external electromagnetic field breaks the WS "gauge" symmetry, so that spin can not be enslaved. The motions then take place along curves: the particle gets localized [6]. New results [21] indicate however that while a WS translation *is unobservable* for an infinite plane wave, it *is observable* for a wave packet. Intuitively, forming and keeping together a wave packet behaves as a sort of interaction.

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